Addition of angular momenta

consider two angular momenta $\vec{J}_1$ and $\vec{J}_2$ which belong to two different subspaces 1 and 2. J_1 and J_2 may refer to two distinct particles of two different properties of the same particle (like $\vec{L}$ and $\vec{S}$ of electron in H atom).

since J_1 and J_2 belong to two distinct spaces $\Rightarrow [\vec{J}_1, \vec{J}_2] = 0$

- for J_1 $\quad J_1^2 |j_1, m_1\rangle = j_1(j_1+1)\hbar^2 |j_1, m_1\rangle$, $\quad J_{1z}|j_1, m_1\rangle = \hbar m_1 |j_1, m_1\rangle$ $\}$ dimension $(2j_1+1)$, $-j_1 \leq m_1 \leq +j_1$

- for J_2 $\quad J_2^2 |j_2, m_2\rangle = j_2(j_2+1)\hbar^2 |j_2, m_2\rangle$, $\quad J_{2z}|j_2, m_2\rangle = \hbar m_2 |j_2, m_2\rangle$ $\}$ dimension $(2j_2+1)$, $-j_2 \leq m_2 \leq +j_2$

the bases $|j_1, m_1\rangle$ are complete and orthogonal

$$\sum_{m_1} |j_1 m_1\rangle\langle j_1 m_1| = 1 \quad \text{and} \quad \langle j_1' m_1' | j_1 m_1\rangle = \delta_{j_1 j_1'}\delta_{m_1 m_1'}$$

similarly for the basis $|j_2, m_2\rangle$

$$\sum_{m_2} |j_2 m_2\rangle\langle j_2 m_2| = 1 \quad \text{and} \quad \langle j_2' m_2' | j_2 m_2\rangle = \delta_{j_2 j_2'}\delta_{m_2 m_2'}$$

- Notice that the four operators $J_1^2, J_{1z}, J_2^2, J_{2z}$ form a complete set of commuting operators (C.S.C.O); they can be diagonalized by the same eigenstates. let us denote them by (shortly $|j_1 m_1; j_2 m_2\rangle$)

$$|j_1, m_1; j_2, m_2\rangle = |j_1, m_1\rangle |j_2 m_2\rangle$$

$$J_1^2 |j_1, m_1; j_2, m_2\rangle = j_1(j_1+1)\hbar^2 |j_1 m_1; j_2 m_2\rangle$$

$$J_{1z}|j_1 m_1 j_2 m_2\rangle = \hbar m_1 |j_1 m_1 j_2 m_2\rangle$$

$$J_2^2 |j_1 m_1 j_2 m_2\rangle = j_2(j_2+1)\hbar^2 |j_1 m_1 j_2 m_2\rangle$$

and $J_{2z}|j_1m_1 j_2m_2\rangle = \hbar m_2 |j_1m_1 j_2m_2\rangle$

the kets $|j_1m_1 j_2m_2\rangle$ form a complete and orthogonal bases

$$\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1m_1 j_2m_2\rangle\langle j_1m_1 j_2m_2| = 1$$

and $\langle j_1'm_1' j_2'm_2' | j_1m_1 j_2m_2\rangle = \delta_{j_1j_1'}\delta_{m_1m_1'}\delta_{j_2j_2'}\delta_{m_2m_2'}$

The concern is now how to find the eigenvalues and the eigenstates of the new set of J^2, J_z, J_1^2, J_2^2 in terms of the old eigenvalues and eigenstates of $J_1^2, J_2^2, J_{1z}, J_{2z}$. the new set of J^2, J_z, J_1^2, J_2^2 form a complete set i.e they can be diagonalized by the same eigenstates. let us denote them by $|j_1, j_2; jm\rangle$. since j_1 and j_2 are usually fixed, we will use a short hand notation ($|jm\rangle$)

$\therefore$ $J_1^2|jm\rangle = j_1(j_1+1)\hbar^2|jm\rangle$

$J_2^2|jm\rangle = j_2(j_2+1)\hbar^2|jm\rangle$

$J^2|jm\rangle = j(j+1)\hbar^2|jm\rangle$

$J_z|jm\rangle = \hbar m|jm\rangle$

the dimension of the combined ~~system~~ space is $(2j_1+1)(2j_2+1)$

so for every j, the # m has $(2j+1)$ allowed values

$$m = \underbrace{-j}_{m_{min}}, -j+1, ----, j-1, \underbrace{j}_{m_{max}} \quad ; \; -j \leq m \leq j$$

of course $|j_1 - j_2| \leq j \leq j_1 + j_2$

notes that the new states $|jm\rangle$ are also complete and orthonormal -

now how can we express the new bases $|j,m\rangle$ in terms of the composite bases $|j_1 m_1 j_2 m_2\rangle$

$$|j,m\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1 m_1 j_2 m_2\rangle \langle j_1 m_1 j_2 m_2 | j m\rangle$$

$$= \sum_{m_1 m_2} \langle j_1 m_1 j_2 m_2 | j m\rangle \; |j_1 m_1 j_2 m_2\rangle$$

$$|j,m\rangle = \sum_{m_1 m_2} C^{jm}_{j_1 m_1 j_2 m_2} \; |j_1 m_1 j_2 m_2\rangle$$

Clebsch - Gordan coefficients (always real)

or can be writtens $|j,m\rangle = \sum_{m_1 m_2} C^{jm}_{j_1 m_1 j_2 m_2} \; |j_1 m_1\rangle |j_2 m_2\rangle$

some properties of $C^{jm}_{j_1 m_1 j_2 m_2}$

1) if $m \neq m_1 + m_2 \Rightarrow \langle j_1 m_1 j_2 m_2 | j m\rangle = 0$
if $m = m_1 + m_2 \Rightarrow \langle j_1 m_1 j_2 m_2 | j m\rangle \neq 0$ finite } Selection Rule

2) if $m_1 = j_1$; $m_2 = j_2$; $j = j_1 + j_2$; $m = j_1 + j_2$ (uper level)

$$\langle j_1 m_1 j_2 m_2 | j m\rangle = \langle j_1 j_1 j_2 j_2 | j m\rangle = 1$$

3) if $m_1 = -j_1$; $m_2 = -j_2$; $j = j_1 + j_2$; $m = -(j_1 + j_2)$ bottom level

$$\langle j_1 m_1 j_2 m_2 | j m\rangle = \langle j_1 j_1 j_2 j_2 | j m\rangle = 1$$

Example: Addition of $j_1 = 1$ and $j_2 = 1$

the total dimension of the combined space $= (2j_1+1)(2j_2+1)$
$= 3 \times 3 = 9$ states

$j_1 + j_2 = 2$; $|j_1 - j_2| = 0$

$\Rightarrow \quad 0 \leq j \leq 2$

$\Rightarrow \quad j = 2, 1, 0 \longrightarrow$ one state $j=0$, $m=0$

5 states
$j=2$
$m = 2, 1, 0, -1, -2$

three states
$j=1$
$m = 1, 0, -1$

a) the subspace $j=2$

upper state is $|2,2\rangle = |1,1,1,1\rangle = |j_1 m_1 j_2 m_2\rangle$

only one way to obtain this state

$\Rightarrow \quad C^{22}_{1111} = 1$

Apply J_- to this state to obtain $|2,1\rangle$

$J_- |2,2\rangle = \sqrt{j(j+1) - m(m-1)}\,\hbar\,|2,1\rangle = 2\hbar\,|2,1\rangle$

$\Rightarrow |2,1\rangle = \frac{1}{2\hbar} J_- |2,2\rangle = \frac{1}{2\hbar}(J_{1-} + J_{2-})|1,1,1,1\rangle$

$= \frac{1}{2\hbar}\left[J_{1-}|1,1,1,1\rangle + J_{2-}|1,1,1,1\rangle\right]$

$j_1 = 1$, $j_2 = 1$

$= \frac{1}{2\hbar}\left[\sqrt{j_1(j_1+1) - m_1(m_1-1)}\,\hbar\,|1,0,1,1\rangle\right.$
$\left.+ \sqrt{j_2(j_2+1) - m_2(m_2-1)}\,\hbar\,|1,1,1,0\rangle\right]$

$= \frac{1}{2\hbar}\left[\sqrt{2}\hbar\,|1,0,1,1\rangle + \sqrt{2}\hbar\,|1,1,1,0\rangle\right]$

$= \frac{1}{\sqrt{2}}\left[|1,0,1,1\rangle + |1,1,1,0\rangle\right]$

$\Rightarrow \quad C^{21}_{1011} = C^{21}_{1110} = \frac{1}{\sqrt{2}}$

apply J_- again to $|2,1\rangle$

$$|2,0\rangle = \frac{1}{\sqrt{6}}\left[\,|1,1,1,-1\rangle + 2\,|1,0,1,0\rangle + |1,-1,1,1\rangle\,\right]$$

$$\Rightarrow C^{20}_{111-1} = \frac{1}{\sqrt{6}} = C^{20}_{1-111} \quad \text{and} \quad C^{20}_{1010} = \frac{2}{\sqrt{6}}$$

similarly $|2,-1\rangle = \frac{1}{\sqrt{2}}\left[\,|1,0,1,-1\rangle + |1,-1,1,0\rangle\,\right]$

$$C^{2-1}_{101-1} = C^{2-1}_{1-110} = \frac{1}{\sqrt{2}}$$

and finally $|2,-2\rangle = |1,-1,1,-1\rangle$; $C^{2-2}_{1-11-1} = 1$

b) the subspace of $j=1$

upper state $|1,1\rangle$ can be obtained by two ways

$|1,1,1,0\rangle$ and $|1,0,1,1\rangle$

$\Rightarrow |1,1\rangle = \alpha\,|1,1,1,0\rangle + \beta\,|1,0,1,1\rangle$ with $\alpha^2+\beta^2=1$

we need another eq^n to calculate α, β completely

from orthogonality of $|1,1\rangle$ with $|2,1\rangle$

$\langle 2,1|1,1\rangle = \frac{\beta}{\sqrt{2}} + \frac{\alpha}{\sqrt{2}} = 0 \Rightarrow \alpha+\beta=0 \Rightarrow \beta=-\alpha$

from $\alpha^2+\beta^2=1 \Rightarrow \alpha^2+\alpha^2=1 \Rightarrow \alpha = \frac{1}{\sqrt{2}}$; $\beta = -\frac{1}{\sqrt{2}}$

$$\Rightarrow |1,1\rangle = \frac{1}{\sqrt{2}}\left[\,|1,1,1,0\rangle - |1,0,1,1\rangle\,\right]$$

$$\Rightarrow C^{11}_{1110} = \frac{1}{\sqrt{2}} \; ; \quad C^{11}_{1011} = -\frac{1}{\sqrt{2}}$$

Now apply J_-

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left[\,|1,1,1,-1\rangle - |1,-1,1,1\rangle\,\right]$$

$$\Rightarrow C^{10}_{111-1} = \frac{1}{\sqrt{2}} \; ; \quad C^{10}_{1-111} = -\frac{1}{\sqrt{2}}$$

and finally $|1,-1\rangle = \frac{1}{\sqrt{2}}\left[\,|1,0,1,-1\rangle - |1,-1,1,0\rangle\,\right]$

$$\Rightarrow C^{1-1}_{101-1} = \frac{1}{\sqrt{2}} \; , \quad C^{1-1}_{1-110} = -\frac{1}{\sqrt{2}}$$

c) the subspace $j=0$, one state $|0,0\rangle$

there are 3 ways to obtain this state

$$|0,0\rangle = a\,|1,1,1,-1\rangle + b\,|1,0,1,0\rangle + c\,|1,-1,1,1\rangle$$

with $a^2+b^2+c^2=1$

This state must be orthogonal to $|2,0\rangle$ and $|1,0\rangle$

thus yielding $a+2b+c=0$ and $a-c=0$

$\Downarrow$ $\Downarrow$

$\leftarrow a=c$

$b+c=0$

$\Rightarrow b=-c$

Now $c^2+c^2+c^2=1 \Rightarrow 3c^2=1 \Rightarrow c=\frac{1}{\sqrt{3}} \Rightarrow a=\frac{1}{\sqrt{3}}\ ;\ b=-\frac{1}{\sqrt{3}}$

$$\Rightarrow |0,0\rangle = \frac{1}{\sqrt{3}}\left[|1,1,1,-1\rangle - |1,0,1,0\rangle + |1,-1,1,1\rangle\right]$$

<u>Example</u>: Addition of $s_1=1/2$ with $s_2=1/2$

or $j_1=1/2$ with $j_2=1/2$

$j = 1, 0 \to$ singlet state $m=0$

$\swarrow$ 3 states (triplet) $m=1,0,-1$

overall we have 4 states

~~two~~ how this work?

for s_1 there are two possibilities $\uparrow_1, \downarrow_1$

for s_2 " " " $\uparrow_2, \downarrow_2$

so $1/2 \otimes 1/2$

$\left.\begin{matrix}\uparrow_1\uparrow_2\\ \downarrow_1\downarrow_2\end{matrix}\right]$ they have intrinsic symmetry

$\left.\begin{matrix}\uparrow_1\downarrow_2\\ \downarrow_1\uparrow_2\end{matrix}\right]$ don't have certain symmetry the last two states can be combined in a way to make one of them symmetric under spin exchange and the other is anti symmetric $(\uparrow_1\downarrow_2 + \downarrow_1\uparrow_2)$ symmetric

$(\uparrow_1\downarrow_2 - \downarrow_1\uparrow_2)$ anti-symmetric

Normalize $\frac{1}{\sqrt{2}}(\uparrow_1\downarrow_2 + \downarrow_1\uparrow_2)$ and $\frac{1}{\sqrt{2}}(\uparrow_1\downarrow_2 - \downarrow_1\uparrow_2)$

so for symmetric states, we have

$$\left.\begin{array}{ll} \uparrow_1\uparrow_2 & S_z = 1 \\ \frac{1}{\sqrt{2}}(\uparrow_1\downarrow_2 + \downarrow_1\uparrow_2) & S_z = 0 \\ \downarrow_1\downarrow_2 & S_z = -1 \end{array}\right\} \begin{array}{l}\text{triplet} \\ S = 1\end{array}$$

$$\frac{1}{\sqrt{2}}(\uparrow_1\downarrow_2 - \downarrow_1\uparrow_2) \quad S_z = 0 \Big\} \begin{array}{l}\text{singlet state} \\ S = 0\end{array}$$

Now some Algebra of two spin 1/2 system

$$\vec{S} = \vec{S_1} + \vec{S_2} \Rightarrow \vec{S}^2 = \vec{S_1}^2 + \vec{S_2}^2 + 2\,\vec{S_1}\cdot\vec{S_2}$$

$$\hbar^2 s(s+1) = s_1(s_1+1)\hbar^2 + s_2(s_2+1)\hbar^2 + 2\,\vec{S_1}\cdot\vec{S_2}$$

$$\hbar^2 s(s+1) = \frac{3}{4}\hbar^2 + \frac{3}{4}\hbar^2 + 2\,\vec{S_1}\cdot\vec{S_2}$$

$$s(s+1) = \frac{3}{2} + \frac{2}{\hbar^2}\vec{S_1}\cdot\vec{S_2}$$

$$= \frac{3}{2} + \frac{2}{\hbar^2}\vec{S_1}\cdot\vec{S_2}$$

$$\Rightarrow \frac{\vec{S_1}\cdot\vec{S_2}}{\hbar^2} = \frac{s(s+1) - 3/2}{2} \; ; \text{ using } \vec{S_1} = \frac{\hbar}{2}\vec{\sigma_1} \; ; \; \vec{S_2} = \frac{\hbar}{2}\vec{\sigma_2}$$

$$\Rightarrow \vec{\sigma_1}\cdot\vec{\sigma_2} = 2s(s+1) - 3 = \begin{cases} -3, & s = 0 \text{ singlet} \\ +1, & s = 1 \text{ triplet} \end{cases}$$

Again, the four states can be written as

$$\chi_+^1 \chi_+^2 , \frac{1}{\sqrt{2}} \left(\chi_+^1 \chi_-^2 + \chi_-^1 \chi_+^2 \right) , \chi_-^1 \chi_-^2 , \frac{1}{\sqrt{2}} \left(\chi_+^1 \chi_-^2 - \chi_-^1 \chi_+^2 \right)$$

so

$$S_z \chi_+^1 \chi_+^2 = (S_{1z} + S_{2z}) \chi_+^1 \chi_+^2 = \frac{\hbar}{2} \chi_+^1 \chi_+^2 + \frac{\hbar}{2} \chi_+^1 \chi_+^2 = \underbrace{\hbar \chi_+^1 \chi_+^2}_{m=+1}$$

$$S_z \chi_-^1 \chi_-^2 = (S_{1z} + S_{2z}) \chi_-^1 \chi_-^2 = \underbrace{-\hbar \chi_-^1 \chi_-^2}_{m=-1}$$

$$S_z \frac{1}{\sqrt{2}} \left(\chi_+^1 \chi_-^2 + \chi_-^1 \chi_+^2 \right) = \frac{1}{\sqrt{2}} \left(\frac{\hbar}{2} \chi_+^1 \chi_-^2 - \frac{\hbar}{2} \chi_-^1 \chi_+^2 - \frac{\hbar}{2} \chi_+^1 \chi_-^2 + \frac{\hbar}{2} \chi_-^1 \chi_+^2 \right)$$

$$= 0 \longrightarrow m = 0$$

finally $S_z \frac{1}{\sqrt{2}} \left(\chi_+^1 \chi_-^2 - \chi_-^1 \chi_+^2 \right) = 0 \longrightarrow m = 0 \quad (s=0)$

- notice that we can move between the states up and down using the raising and lowering operators $S_\pm$

$$S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} ; \quad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ; \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$S_- \chi_+ = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hbar \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar \chi_- \quad \therefore S_- \chi_+ = \hbar \chi_-$$

and $$S_- \chi_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \quad \therefore S_- \chi_- = 0$$

$$|s, m\rangle = \begin{cases} \text{triplet} & |1,1\rangle, |1,0\rangle, |1,-1\rangle \\ \text{singlet} & |0,0\rangle \end{cases}$$

start from the upper level $|1,1\rangle$ and apply S_- ;

where $S_- |s, m\rangle = \hbar \sqrt{s(s+1) - m(m-1)} \, |s, m-1\rangle$

$$\therefore \quad S_-|1,1\rangle = \sqrt{2}\,\hbar\,|1,0\rangle \quad \Rightarrow |1,0\rangle = \frac{1}{\sqrt{2}\,\hbar} S_-|1,1\rangle$$

Now need to find $S_-|1,1\rangle$

$$S_-|1,1\rangle = (S_{1-} + S_{2-})\,\chi_+^1 \chi_+^2 = (S_{1-}\chi_+^1)\chi_+^2 + \chi_+^1 (S_{2-}\chi_+^2)$$

$$= \hbar\, \chi_-^1 \chi_+^2 + \hbar\, \chi_+^1 \chi_-^2$$

$$\downarrow$$

$$\chi_+^1 \chi_+^2$$

$$= \frac{\sqrt{2}}{\sqrt{2}}\hbar\,(\chi_-^1 \chi_+^2 + \chi_+^1 \chi_-^2)$$

$$= \sqrt{2}\,\hbar\, \underbrace{\frac{\chi_-^1 \chi_+^2 + \chi_+^1 \chi_-^2}{\sqrt{2}}}$$

$$\therefore \quad |1,0\rangle = \frac{\chi_-^1 \chi_+^2 + \chi_+^1 \chi_-^2}{\sqrt{2}} = \frac{\downarrow_1 \uparrow_2 + \uparrow_1 \downarrow_2}{\sqrt{2}}$$

similarly $\quad S_-|1,0\rangle = \sqrt{2}\,\hbar\,|1,-1\rangle$

$$\Rightarrow |1,-1\rangle = \frac{1}{\sqrt{2}\,\hbar} S_-|1,0\rangle$$

Now $S_-|1,0\rangle = S_-\left(\frac{\chi_-^1 \chi_+^2 + \chi_+^1 \chi_-^2}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(S_{1-} + S_{2-})\left[\chi_-^1 \chi_+^2 + \chi_+^1 \chi_-^2\right]$

$$= \frac{1}{\sqrt{2}}\left[\hbar\chi_-^1 \chi_-^2 + \hbar\chi_-^1\chi_-^2\right.$$

$$= \frac{2\hbar}{\sqrt{2}}\chi_-^1 \chi_-^2 = \sqrt{2}\,\hbar\,\chi_-^1 \chi_-^2$$

$$\Rightarrow |1,-1\rangle = \chi_-^1 \chi_-^2 = \downarrow_1 \downarrow_2$$

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finally we can not use S_- to find the singlet state as it is unique and lies outside the triplet states.

$$S_-\left(\frac{\chi_-^1 \chi_+^2 - \chi_+^1 \chi_-^2}{\sqrt{2}}\right) = 0$$